

On Davis-Putnam reductions for minimally unsatisfiable clause-sets

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Abstract. DP-reduction $F \rightsquigarrow \text{DP}_v(F)$, applied to a clause-set F and a variable v , replaces all clauses containing v by their resolvents (on v). A basic case, where the number of clauses is decreased (i.e., $c(\text{DP}_v(F)) < c(F)$), is *singular DP-reduction* (sDP-reduction), where v must occur in one polarity only once. For minimally unsatisfiable $F \in \mathcal{MU}$, sDP-reduction produces another $F' := \text{DP}_v(F) \in \mathcal{MU}$ with the same deficiency, that is, $\delta(F') = \delta(F)$; recall $\delta(F) = c(F) - n(F)$, using $n(F)$ for the number of variables. Let $\text{sDP}(F)$ for $F \in \mathcal{MU}$ be the set of results of complete sDP-reduction for F ; so $F' \in \text{sDP}(F)$ fulfil $F' \in \mathcal{MU}$, are *non-singular* (every literal occurs at least twice), and we have $\delta(F') = \delta(F)$. We show that for $F \in \mathcal{MU}$ all complete reductions by sDP must have the same length, establishing the *singularity index* of F . In other words, for $F', F'' \in \text{sDP}(F)$ we have $n(F') = n(F'')$. In general the elements of $\text{sDP}(F)$ are not even (pairwise) isomorphic. Using the fundamental characterisation by Kleine Büning, we obtain as application of the singularity index, that we have *confluence modulo isomorphism* (all elements of $\text{sDP}(F)$ are pairwise isomorphic) in case $\delta(F) = 2$. In general we prove that we have confluence (i.e., $|\text{sDP}(F)| = 1$) for saturated F (i.e., $F \in \mathcal{SMU}$). More generally, we show confluence modulo isomorphism for *eventually saturated* F , that is, where we have $\text{sDP}(F) \subseteq \mathcal{SMU}$, yielding another proof for confluence modulo isomorphism in case of $\delta(F) = 2$.

1 Introduction

Minimally unsatisfiable clause-sets (“MU’s”) are a fundamental form of irredundant unsatisfiable clause-sets. Regarding the subset relation, they are the hardest examples for proof systems. A substantial amount of insight has been gained into their structure, as witnessed by the handbook article [6]. A related area of MU, which gained importance in recent industrial applications, is the study of “MUS’s”, that is minimally unsatisfiable *sub*-clause-sets $F' \in \mathcal{MU}$ with $F' \subseteq F$ as the “cores” of unsatisfiable clause-sets F ; see [15] for a recent overview. For the investigations of this paper there are two main sources: The structure of MU (see Subsection 1.1), and the study of DP-reduction as started with [7,12,13]:

- A fundamental result shown there is that DP-reduction is commutative modulo subsumption (see Subsection 5.2 for the precise formulation).
- Singular DP-reduction is a special case of length-reducing DP-reduction (while in general one step of DP-reduction can yield a quadratic blow-up).

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- Confluence *modulo isomorphism* was shown in [7] (Theorem 13, Page 52) for a combination of subsumption elimination with special cases of length-reducing DP-reductions, namely DP-reduction in case no (non-tautological) resolvent is possible, and singular DP-reduction in case there is only one side clause, or the main clause is of length at most 2 (see Definition 4).

The basic questions for this paper are:

- When does singular DP-reduction, applied to MU, yield unique (non-singular) results (i.e., we have confluence)?
- And when are the results at least determined up to isomorphism (i.e., we have confluence modulo isomorphism)?

Different from the result from [7] mentioned above, we do not consider restricted versions of singular DP-reduction, but we restrict the class of clause-sets to which singular DP-reduction is applied (namely to subclasses of MU).

1.1 Investigations into the structure of $\mathcal{MU}(k)$

We give now a short overview on the problem of classifying $F \in \mathcal{MU}$ in terms of the deficiency $\delta(F) := c(F) - n(F)$, that is, the problem of characterising the levels $\mathcal{MU}_{\delta=k} := \{F \in \mathcal{MU} : \delta(F) = k\}$ (due to greater expressivity and generality, we prefer this notation over $\mathcal{MU}(k)$); see [6] for further information.

The field of the combinatorial study of minimally unsatisfiable clause-sets was opened by [1], showing the fundamental insight $\delta(F) \geq 1$ for $F \in \mathcal{MU}$ (see [9,6] for generalisations of the underlying method, based on autarky theory). Also $\mathcal{SMU}_{\delta=1}$ was characterised there, where $\mathcal{SMU} \subset \mathcal{MU}$ is the set of “saturated” minimally unsatisfiable clause-sets, which are minimal not only w.r.t. having no superfluous clauses, but also w.r.t. that no clause can be further weakened. The fundamental “saturation method” $F \in \mathcal{MU} \rightsquigarrow F' \in \mathcal{SMU}$ was introduced in [4] (see Definition 1). Basic for all studies of MU is detailed knowledge on minimal number of occurrences of a (suitable) variable (yielding a suitable splitting variable): see [14] for the current state-of-art. The levels $\mathcal{MU}_{\delta=k}$ are decidable in polynomial time by [3,8]; see [16,10] for further extensions.

“Singular” variables v in $F \in \mathcal{MU}$, that is, variables occurring in at least one polarity only once, play a fundamental role — they are degenerations which (usually) need to be eliminated by *singular DP-reduction*. Let $\mathcal{MU}' \subset \mathcal{MU}$ be the set of non-singular minimally unsatisfiable clause-sets (not having singular variables), that is, the results of applying singular DP-reduction to the elements of \mathcal{MU} as long as possible. The fundamental problem is the characterisation of $\mathcal{MU}'_{\delta=k}$ for arbitrary $k \in \mathbb{N}$. Up to now only $k \leq 2$ has been solved (see Subsection 3.3 for some details): $\mathcal{MU}'_{\delta=1}$ has been determined in [2], while $\mathcal{MU}'_{\delta=2} = \mathcal{SMU}'_{\delta=2}$ has been determined in [5]. Regarding higher deficiencies, until now only (very) partial results in [17] exist. Regarding singular minimally unsatisfiable clause-sets, also $\mathcal{MU}_{\delta=1}$ is very well known (with further extensions and generalisations in [8], and generalised to non-boolean clause-sets in [11]), while for $\mathcal{MU}_{\delta=2}$ not much is known (Subsection 6.1 provides first insights).

For characterising $\mathcal{MU}'_{\delta=k}$, we need (very) detailed insights into (arbitrary) $\mathcal{MU}_{\delta < k}$. Assuming that we know $\mathcal{MU}'_{\delta < k}$, such insights can be based on some classification of $F \in \mathcal{MU}_{\delta < k}$ obtained from the set $\text{sDP}(F) \subseteq \mathcal{MU}'_{\delta < k}$ of singular-DP-reduction results. The easiest case is when $|\text{sDP}(F)| = 1$ holds

(confluence), the second-easiest case is where all elements of $\text{sDP}(F)$ are pairwise isomorphic. This is the basic motivation for the questions raised and partially solved in this article. For general k we have no conjecture yet how the classification of $\mathcal{MU}'_{\delta=k}$ could look like (besides the basic conjecture that enumeration of the isomorphism types can be done efficiently). However for unsatisfiable hitting clause-sets we have the conjecture stated in [14], that for every $k \in \mathbb{N}$ there are only finitely many isomorphism types in $\mathcal{UHT}_{\delta=k}$.

1.2 Overview on results

Section 3 introduces the basic notions regarding singularity, and the basic characterisations of singular DP-reduction on minimally unsatisfiable clause-sets are given in Subsection 3.2. In Section 4 we consider the question of confluence of singular DP-reduction, with the first main result Theorem 16, showing confluence for saturated clause-sets. Section 5 mainly considers the question of changing the order of DP-reductions without changing the result. The second main result of this article is Theorem 39, establishing the singularity index. Section 6 is devoted to show confluence modulo isomorphism on eventually saturated clause-sets (Theorem 44), the third main result. As an application we determine the “types” of (possibly singular) minimally unsatisfiable clause-sets of deficiency 2 via Theorem 47. We conclude with a collection of open problems in Section 7.

1.3 Applications

Our current main application, which motivated the questions tackled in this paper in the first place, is the project of classifying the structure of $\mathcal{MU}_{\delta=k}$ as discussed in Subsection 1.1: Knowing some form of invariance of singular DP-reduction enables one to classify also *singular* minimally unsatisfiable clause-sets, based on knowing the non-singular minimally unsatisfiable clause-sets of the same deficiency; see Subsection 6.1 for a first example.

For worst-case upper bounds of SAT decision (or related problems) we sometimes need to guarantee that certain reductions will yield a certain decrease in some parameter, for example the number of variables, independently of the special order of reductions — this is exactly established for singular DP-reduction by the singularity index (using Corollary 40).

Finally, singular DP-reduction is a very basic and efficient reduction, which should be helpful in the search for MUS's, using that a singular variable for F is also singular for $F' \subseteq F$ with $F' \in \mathcal{MU}$. The basic results of Section 3 make it possible to control the effects of singular DP-reduction, while our main results enable one to estimate the inherent non-determinism.

2 Preliminaries

We follow the general notations and definitions as outlined in [6]. Complementation of literals x is denoted by \overline{x} , while for a set L of literals we define $\overline{L} := \{\overline{x} : x \in L\}$. A **clause** C is a finite and clash-free set of literals (i.e., $C \cap \overline{C} = \emptyset$), while a **clause-set** $F \in \mathcal{CLS}$ is a finite set of clauses. We denote by $\text{var}(F)$ the set of (occurring) variables, by $n(F) := |\text{var}(F)|$ the number of variables, by $c(F) := |F|$ the number of clauses, and finally by $\delta(F) := c(F) - n(F)$

the deficiency. For clause-sets F, G we denote by $F \cong G$ that both clause-sets are **isomorphic**, that is, the variables of F can be renamed and potentially flipped so that F is turned into G ; more precisely, an isomorphism α from F to G is a bijection α on literal-sets which preserves complementation and which maps the clauses of F precisely to the clauses of G . The *literal-degree* $\text{ld}_F(x) \in \mathbb{N}_0$ of a literal x for a clause-set F is the number of clauses the literal appears in, i.e., $\text{ld}_F(x) := |\{C \in F : x \in C\}|$. The *variable-degree* $\text{vd}_F(v) \in \mathbb{N}_0$ for a variable v is the number of clauses the variable appears in, i.e., $\text{vd}_F(v) := \text{ld}_F(v) + \text{ld}_F(\bar{v})$.

For a clause-set F and a variable v by $\mathbf{DP}_v(F)$ we denote the result of applying DP-reduction on v , that is, removing all clauses containing v and adding all resolvents on v . More formally

$$\mathbf{DP}_v(F) := \{C \in F : v \notin \text{var}(C)\} \cup \{C \diamond D : C, D \in F, C \cap \bar{D} = \{v\}\},$$

where clauses C, D are resolvable iff they clash in exactly one literal, i.e., iff $|C \cap \bar{D}| = 1$, while for resolvable clauses C, D the resolvent $C \diamond D := (C \cup D) \setminus \{x, \bar{x}\}$ for $C \cap \bar{D} = \{x\}$ is defined as the union minus the resolution literals (the two clashing literals). $\mathbf{DP}_v(F)$ is logically equivalent to the existential quantification of F by v , and thus F and $\mathbf{DP}_v(F)$ are satisfiability-equivalent.

The set of minimally unsatisfiable clause-sets is $\mathcal{MU} \subset \mathcal{CLS}$, the set of all clause-sets which are unsatisfiable, while removal of any clause makes them satisfiable. Furthermore the set of saturated minimally unsatisfiable clause-sets is $\mathcal{SMU} \subset \mathcal{MU}$, which is the set of minimally unsatisfiable clause-sets such that addition of any literal to any clause renders them satisfiable. Note that for $v \in \text{var}(F)$ with $F \in \mathcal{MU}$ we have $\text{vd}_F(v) \geq 2$. We recall the fact ([4,14]) that every minimally unsatisfiable clause-set $F \in \mathcal{MU}$ can be **saturated**, i.e., by adding literal occurrences to F we obtain $F' \in \mathcal{SMU}$ with $\text{var}(F') = \text{var}(F)$ such that there is a bijection $\alpha : F \rightarrow F'$ with $C \subseteq \alpha(C)$ for all $C \in F$.

Definition 1. The operation $\mathbf{S}(F, C, x) := (F \setminus \{C\}) \cup (C \cup \{x\}) \in \mathcal{CLS}$ (adding literal x to clause C in F) is defined if $F \in \mathcal{CLS}$, $C \in F$, and x is a literal with $\text{var}(x) \in \text{var}(F) \setminus \text{var}(C)$. A **saturation** $F' \in \mathcal{SMU}$ of $F \in \mathcal{MU}$ is obtained by a sequence $F = F_0, \dots, F_m = F'$, $m \in \mathbb{N}_0$, such that for $0 \leq i < m$ there are C_i, x_i with $F_{i+1} = \mathbf{S}(F_i, C_i, x_i)$, such that for all $1 \leq i \leq m$ we have $F_i \notin \mathcal{SAT}$, and such that the sequence cannot be extended. Note that $n(F') = n(F)$ and $c(F') = c(F)$ holds (and thus $\delta(F') = \delta(F)$). More generally, a **partial saturation** of a clause-set $F \in \mathcal{MU}$ is a clause-set $F' \in \mathcal{MU}$ such that $\text{var}(F') = \text{var}(F)$ and there is a bijection $\alpha : F \rightarrow F'$ such that for all $C \in F$ we have $C \subseteq \alpha(C)$.

A clause-set F is **hitting** (as DNF known as “disjoint” or “orthogonal”) if every two different clauses clash in at least one literal; the set of hitting clause-sets is denoted by $\mathcal{HIT} \subset \mathcal{CLS}$, the set of unsatisfiable hitting clause-sets by $\mathcal{UHIT} := \mathcal{HIT} \cap \mathcal{USAT}$. Obviously we have $\mathcal{UHIT} \subset \mathcal{SMU}$. The following (new) observation is fundamental for the study of hitting clause-sets:

Lemma 2. For $F \in \mathcal{HIT}$ and a variable v we have $\mathbf{DP}_v(F) \in \mathcal{HIT}$.

Proof. Consider clauses $E_1, E_2 \in \mathbf{DP}_v(F)$, $E_1 \neq E_2$. If $E_1, E_2 \in F$, then E_1, E_2 clash since F is hitting. The two remaining cases are (w.l.o.g.) $E_1 \in F, E_2 \notin F$ and $\bar{E}_1, E_2 \notin F$. In the first case assume $E_2 = C_2 \diamond D_2$ for $C_2, D_2 \in F$ with $C_2 \cap \bar{D}_2 = \{v\}$. Since $v \notin \text{var}(E_1)$, it clashes E_1 with C_2 (as well as with D_2) and thus with E_2 . For the second case also assume $E_1 = C_1 \diamond D_1$ for $C_1, D_1 \in F$ with $C_1 \cap \bar{D}_1 = \{v\}$. We must have $C_1 \neq C_2$ or $D_1 \neq D_2$, yielding a clash between C_1, C_2 resp. D_1, D_2 , and thus also E_1, E_2 clash. \square

Corollary 3. *For $F \in \mathcal{UHT}$ and a variable v we have $\text{DP}_v(F) \in \mathcal{UHT}$.*

3 Singularity

In this section we present basic results on singular variables in minimally unsatisfiable clause-sets. Lemmas 6, 9 yield basic characterisations of singular DP-reduction for minimally unsatisfiable resp. saturated minimally unsatisfiable clause-sets. None of the results are difficult, but the value of these considerations are in the choice of concepts and the presentations of the somewhat subtle facts.³ We conclude in Subsection 3.3 by the known classifications of non-singular minimally unsatisfiable clause-sets of deficiencies 1, 2.

3.1 Singular variables

Definition 4. *We call a variable v **singular** for a clause-set $F \in \mathcal{CLS}$ if we have $\min(\text{ld}_F(v), \text{ld}_F(\bar{v})) = 1$; the set of singular variables of F is denoted by $\text{var}_s(F) \subseteq \text{var}(F)$. F is called **nonsingular** if F does not contain singular variables. Furthermore we use the following notations:*

- $\mathcal{MU}' := \{F \in \mathcal{MU} : \text{var}_s(F) = \emptyset\}$ denotes the set of nonsingular MU's;
- $\mathcal{SMU}' := \mathcal{SMU} \cap \mathcal{MU}'$ is the set of nonsingular saturated MU's;
- $\mathcal{UHT}' := \mathcal{UHT} \cap \mathcal{SMU}' = \mathcal{HT} \cap \mathcal{MU}'$ is the set of nonsingular unsatisfiable hitting clause-sets.

More precisely, we call v **m -singular** for F for some $m \in \mathbb{N}$, if v is singular for F with $m = \text{vd}_F(v) - 1$. The set of 1-singular variables of F is denoted by $\text{var}_{1s}(F) \subseteq \text{var}_s(F)$. That a variable is m -singular for some $m \geq 2$ is simply called **non-1-singular** (so “non-1-singular” variables are singular); the set of non-1-singular variables of F is denoted by $\text{var}_{\neg 1s}(F) := \text{var}_s(F) \setminus \text{var}_{1s}(F)$. A **singular literal** for a singular variable v is a literal x with $\text{var}(x) = v$ and $\text{ld}_F(x) = 1$; if the underlying variable is 1-singular, then some choice is applied, so that we can speak of “the” singular literal of a singular variable. For a singular literal x we call the clause $C \in F$ with $x \in C$ the **main clause**, while the **side clauses** are the clauses $D_1, \dots, D_m \in F$ with $\bar{x} \in D_i$ (here v is m -singular).

3.2 Singular DP-reduction

The following special application of DP-reduction appears at many places in the literature (see [5], or Appendix B in [8] and subsequent [16,10]), and is fundamental for investigations of minimally unsatisfiable clause-sets:

Definition 5. *A **singular DP-reduction** is a reduction $F \rightsquigarrow \text{DP}_v(F)$, where v is singular for $F \in \mathcal{MU}$. For $F, F' \in \mathcal{MU}$ by $F \xrightarrow{sDP} F'$ we denote that F' is obtained from F by one step of singular DP-reduction; i.e., there is a singular variable v for F with $F' = \text{DP}_v(F)$, where v is called the **reduction variable**. And we write $F \xrightarrow{sDP}_* F'$ if F' is obtained from F by an arbitrary number of steps (possibly zero) of singular DP-reductions. The set of all nonsingular clause-sets obtainable from F by singular DP-reduction is denoted by $\text{sDP}(F)$:*

$$\text{sDP}(F) := \{F' \in \mathcal{MU}' : F \xrightarrow{sDP}_* F'\}.$$

³ Various of these facts the first author discussed 8 years ago with Stefan Szeider.

The following lemma is kind of “folklore”, but apparently the only place where its assertions are (partially) stated in the literature (in a more general form) is [10], Lemma 6.1 (we add here various details):

Lemma 6. *Consider a clause-set F and a singular variable v for F . Then the following assertions are equivalent:*

1. F is minimally unsatisfiable.
2. $\delta(\text{DP}_v(F)) = \delta(F)$ and $\text{DP}_v(F)$ is minimally unsatisfiable.
3. $\text{DP}_v(F)$ is minimally unsatisfiable, and for the main clause C and the side clauses D_1, \dots, D_m for v (in F) we have:
 - (a) Every D_i clashes with C in exactly one variable (namely in v).
 - (b) For $1 \leq i < j \leq m$ we have $C \diamond D_i \neq C \diamond D_j$.
 - (c) For $E \in F$ with $v \notin \text{var}(E)$ and for all $1 \leq i \leq m$ we have $C \diamond D_i \neq E$.

Proof. The equivalence of Part 1 and Part 2 is a special case of Lemma 6.1 in [10]. Part 2 implies Part 3, since if one of the conditions 3a, 3b or 3c would not hold, then the deficiency of $\text{DP}_v(F)$ would be (strictly) smaller than F , contradicting the assumption $\delta(\text{DP}_v(F)) = \delta(F)$. Finally we show that Part 3 implies Part 1. Since $\text{DP}_v(F)$ is minimally unsatisfiable, F is unsatisfiable. Now suppose that F is not minimally unsatisfiable. So for some clause $E \in F$ the clause-set $F' := F \setminus \{E\}$ is still unsatisfiable. By condition 3a we know that $C \diamond D_i$ must be in $\text{DP}_v(F)$ for all $i \in \{1, \dots, m\}$. Thus clause E can not be the main clause C , and if $m = 1$, then E can not be the side clause neither. So v is still a singular variable in F' . Since $\text{DP}_v(F)$ is minimally unsatisfiable, while we have $\text{DP}_v(F') \subseteq \text{DP}_v(F)$, we obtain $\text{DP}_v(F') = \text{DP}_v(F)$, that is, either E is one of the side clauses and its resolvent with C was obtained by some other resolution or was already present, or E does not contain v , and thus E must be a resolvent. In any case we get a contradiction with one of 3b or 3c. \square

Corollary 7. *If $F \in \mathcal{MU}$ and v is a singular variable of F , then also $\text{DP}_v(F) \in \mathcal{MU}$, where $\delta(\text{DP}_v(F)) = \delta(F)$. So the classes $\mathcal{MU}_{\delta=k}$ for $k \in \mathbb{N}$ are stable under singular DP-reduction.*

Corollary 8. *Consider $F \in \mathcal{MU}$ and a singular variable v with singular literal x , with main clause C and side clauses D_1, \dots, D_m . Then adding $C \setminus \{x\}$ to D_i for all $i \in \{1, \dots, m\}$ is a partial saturation of F (recall Definition 1).*

Lemma 6 can be strengthened for saturated F by requiring special conditions for the occurrences of the singular variable.

Lemma 9. *Consider a clause-set F and a singular variable v for F . For a singular literal x for v consider the main clause C and the side clauses $D_1, \dots, D_m \in F$. Let $C' := C \setminus \{x\}$ and $D'_i := D_i \setminus \{\bar{x}\}$. The following assertions are equivalent:*

1. F is saturated minimally unsatisfiable.
 2. The following three conditions hold:
 - (a) $\text{DP}_v(F)$ is saturated minimally unsatisfiable;
 - (b) $C' = \bigcap_{i=1}^m D'_i$;
 - (c) for every $E \in F$ with $v \notin \text{var}(E)$ we have $C' \not\subseteq E$.
- Note that conditions 2b, 2c together imply the condition that for $E \in F$ we have $C' \subseteq E$ if and only if $v \in \text{var}(E)$ holds.*

Proof. First assume that F is saturated minimally unsatisfiable. If there would be $E \in F$ with $v \notin \text{var}(F)$ and $C' \subseteq E$, then for $F' := S(F, E, \bar{v})$ we had $\text{DP}_v(F') = \text{DP}_v(F)$, and thus F' would be unsatisfiable, contradicting saturation of F . We have $C' \subseteq \bigcap_{i=1}^m D'_i$, since if there were a literal $y \in C'$ and $y \notin D'_i$ for some i , then $\text{DP}_v(S(F, D_i, y)) = \text{DP}_v(F)$. And we have $C' \supseteq \bigcap_{i=1}^m D'_i$, since if there were a literal y contained in all D'_i , but not in C' , then $\text{DP}_v(S(F, C, y)) = \text{DP}_v(F)$.

By Lemma 6 we know that $\text{DP}_v(F)$ is minimally unsatisfiable, and that all resolutions are carried out, with no contraction due to coinciding resolvents or coincidence of a resolvent with an existing clause. Assume that $\text{DP}_v(F)$ is not saturated, that is, there is a clause E and a literal y with $G := S(\text{DP}_v(F), E, y) \in \mathcal{USAT}$. If $E \in F$ then $\text{DP}_v(S(F, E, y)) = G \in \mathcal{USAT}$, and so there is some $1 \leq i \leq m$ with $E = C \diamond D_i$. But now $\text{DP}_v(S(F, D_i, y)) = G$, yielding a contradiction.

Now we consider the opposite direction, that is, we assume that $C' = \bigcap_{i=1}^m D'_i$, that $\text{DP}_v(F)$ is saturated minimally unsatisfiable, and that C' is contained in some clause of F iff this clause contains the variable v . First we establish the three conditions from Lemma 6, Part 3. Since clauses are clash-free, C' has no conflict with any D'_i , and thus the clash-freeness-condition is fulfilled. If we had $C \diamond D_i = C \diamond D_j$ for $i \neq j$, then w.l.o.g. there must be a literal $y \in C'$ with $y \in D'_i$ and $y \notin D'_j$, which is impossible since C' contains only literals which are common to all side clauses. Finally, since all resolvents $C \diamond D_i$ subsume the parent clause D_i , by the minimal unsatisfiability of F also Condition 3c is fulfilled. So we have established that F is minimally unsatisfiable.

Assume that F is not saturated, that is, there exists a clause $E \in F$ and a literal y with $G := S(F, E, y) \in \mathcal{MU}$. Let $F' := \text{DP}_v(F)$ and $G' := \text{DP}_v(G)$ (note $G' \in \mathcal{USAT}$, and that $F' \in \mathcal{SMU}$ by assumption). Our strategy is to derive a contradiction by showing that literal occurrences can be added to F' in such a way that G' is obtained, contradicting that F' is saturated.

First consider $E \notin \{C\} \cup \{D_i\}_{1 \leq i \leq m}$. If $\text{var}(y) \neq v$, then $G' = S(F', E, y)$. If $y = \bar{v}$, then $G' = S(F', \{E\}, C')$ (using Condition 2c). It remains the case $y = v$, but this case is impossible since then for all $1 \leq i \leq m$ we have $C \diamond D_i = D'_i \subseteq (E \cup \{v\}) \diamond D_i = E \cup D'_i$, and thus $\text{DP}_v(G)$ would be satisfiability equivalent to $\text{DP}_v(G \setminus \{E \cup \{v\}\})$, whence G would not be minimally unsatisfiable.

So we have $E \in \{C\} \cup \{D_i\}_{1 \leq i \leq m}$, i.e., $v \in \text{var}(C)$. If $E = C$, then $G' = S(F', \{D'_i\}_{1 \leq i \leq m}, y)$, using that C' is the intersection all the D'_i , and thus at least one D'_i does not contain y . And if $E = C_i$ for some i , then $G' = S(F', D'_i, y)$. \square

Corollary 10. *The class \mathcal{SMU} is stable under singular DP-reduction.*

3.3 On the classification of MU for deficiencies up to 2

In [2] it is shown that every $F \in \mathcal{MU}_{\delta=1}$ contains a 1-singular variable (see [8,14] for further generalisations). Thus by Corollary 7 we get that singular DP-reduction on $\mathcal{MU}_{\delta=1}$ must end in $\{\{\perp\}\}$, and we have $\mathcal{MU}'_{\delta=1} = \{\{\perp\}\}$. Also $\mathcal{MU}'_{\delta=2}$ is completely known:

Definition 11. *Consider $n \geq 2$, let addition for the indices of variables v_1, \dots, v_n be understood modulo n (so $n+1 \rightsquigarrow 1$), and define $P_n := \{v_1, \dots, v_n\}$, $N_n := \{\bar{v}_1, \dots, \bar{v}_n\}$, $C_i := \{\bar{v}_i, v_{i+1}\}$ for $i \in \{1, \dots, n\}$, and finally $\mathcal{F}_n := \{P_n, N_n\} \cup \{C_i : i \in \{1, \dots, n\}\} \in \mathcal{MU}'_{\delta=2}$.*

So $n(\mathcal{F}_n) = n$ and $c(\mathcal{F}_n) = n + 2$. For example

$$\begin{aligned}\mathcal{F}_2 &= \{\{v_1, v_2\}, \{\overline{v_1}, \overline{v_2}\}, \{\overline{v_1}, v_2\}, \{\overline{v_2}, v_1\}\} \\ \mathcal{F}_3 &= \{\{v_1, v_2, v_3\}, \{\overline{v_1}, \overline{v_2}, \overline{v_3}\}, \{\overline{v_1}, v_2\}, \{\overline{v_2}, v_3\}, \{\overline{v_3}, v_1\}\}.\end{aligned}$$

Theorem 12. [5] For $F \in \mathcal{MU}'_{\delta=2}$ we have $F \cong \mathcal{F}_{n(F)}$.

4 Confluence of singular DP-reduction

In this section we introduce the question of confluence of singular DP-reduction. In Subsection 4.1 we define “confluence” and “confluence modulo isomorphism”, and discuss basic examples. In Subsection 4.2 we obtain our first major result, namely confluence for \mathcal{SMU} (Theorem 16).

4.1 The question of confluence

Definition 13. Let \mathcal{CFMU} be the set of $F \in \mathcal{MU}$ where singular DP-reduction is confluent, and let \mathcal{CFIMU} be the set of $F \in \mathcal{MU}$ where singular DP-reduction is confluent modulo isomorphism:

$$\begin{aligned}\mathcal{CFMU} &:= \{F \in \mathcal{MU} \mid |\text{sDP}(F)| = 1\} \\ \mathcal{CFIMU} &:= \{F \in \mathcal{MU} \mid \forall F', F'' \in \text{sDP}(F) : F' \cong F''\}.\end{aligned}$$

In Subsection 3.3 we mentioned that $\mathcal{MU}_{\delta=1} \subseteq \mathcal{CFMU}$ holds. The following example shows $\mathcal{MU}_{\delta=2} \not\subseteq \mathcal{CFMU}$: Let $F \in \mathcal{MU}_{\delta=2}$ be obtained from \mathcal{F}_2 by “inverse singular DP-reduction”, adding a new singular variable v and replacing the two clause $\{v_1, v_2\}, \{v_1, \overline{v_2}\} \in \mathcal{F}_2$ by the three clauses $\{v, v_1\}, \{\overline{v}, v_2\}, \{\overline{v}, \overline{v_2}\}$, obtaining F (the other two clauses in F are $\{\overline{v_1}, v_2\}, \{\overline{v_1}, \overline{v_2}\}$). Singular DP-reduction on v yields \mathcal{F}_2 (and thus by Lemma 6 we get indeed $F' \in \mathcal{MU}_{\delta=2}$). The second singular variable of F is v_1 , and sDP-reduction on v_1 yields $F' := \{\{v, v_2\}, \{v, \overline{v_2}\}, \{\overline{v}, v_2\}, \{\overline{v}, \overline{v_2}\}\}$, where $F' \neq \mathcal{F}_2$. Note however that we have $F' \cong \mathcal{F}_2$, and in Theorem 47 we will indeed see that we have $\mathcal{MU}_{\delta=2} \subseteq \mathcal{CFIMU}$.

Finally we show $\mathcal{MU}_{\delta=3} \not\subseteq \mathcal{CFIMU}$, constructing $F \in \mathcal{MU}_{\delta=3}$ with $\text{sDP}(F) = \{F_1, F_2\}$ where $F_1 \not\cong F_2$. Let $G_1 := \mathcal{F}_2$, and let G_2 be the variable-disjoint copy of G_1 obtained by replacing variables v_1, v_2 with v'_1, v'_2 . Let w be a new variable, and obtain F_1 by “full gluing” of G_1, G_2 on w , that is, add literal w to all clauses of G_1 , add literal \overline{w} to all clauses of G_2 , and let F_0 be the union of these two clause-sets. Obviously we have $F_1 \in \mathcal{UHT}'_{\delta=3}$. Finally obtain F from F_1 by inverse singular DP-reduction, adding a new (singular) variable v , and replacing the two clauses $\{w, v_1, v_2\}, \{w, v_1, \overline{v_2}\}$ by the three clauses $\{v, w, v_1\}, \{\overline{v}, v_2\}, \{\overline{v}, w, \overline{v_2}\}$. Singular DP-reduction on v yields F_1 , and thus $F \in \mathcal{MU}_{\delta=3}$. The second singular variable of F is v_1 , and sDP-reduction on v_1 yields a clause-set F_2 containing one binary clause, since we left out w in the replacement-clause $\{\overline{v}, v_2\}$. Since all clauses in F_1 have length 3, we see $F_2 \not\cong F_1$.

4.2 Confluence on saturated \mathcal{MU}

Definition 14. For clause-sets F, G we write $F \subseteq^{\rightarrow} G$ if for all $C \in F$ there is $D \in G$ with $C \subseteq D$.

If $F \subseteq^{\mapsto} G$, then we say that “ F is a subset of $G \bmod(\text{ulo})$ supersets”. \subseteq^{\mapsto} is a quasi-order on arbitrary clause-sets and a partial order on subsumption-free clause-sets, and thus \subseteq^{\mapsto} is a partial order on \mathcal{MU} . The minimal element of \subseteq^{\mapsto} on \mathcal{CLS} is \top , the minimal element on \mathcal{MU} is $\{\perp\}$. Now we show that “nonsingular saturated patterns” are not destroyed by singular DP-reduction:

Lemma 15. *Consider $F_0, F, F' \in \mathcal{MU}$ with $F \xrightarrow{\text{sDP}}_* F'$.*

1. *If F_0 is nonsingular, then $F_0 \subseteq^{\mapsto} F \Rightarrow F_0 \subseteq^{\mapsto} F'$.*
2. *If $F_0, F, F' \in \mathcal{SMU}$, then $F_0 \subseteq^{\mapsto} F' \Rightarrow F_0 \subseteq^{\mapsto} F$.*

Proof. W.l.o.g. we can assume for both parts that $F' = \text{DP}_v(F)$ for a singular variable v of F . Part 1 follows from the facts that $v \notin \text{var}(F_0)$ due to the nonsingularity of F_0 , and that due to the minimal unsatisfiability of F no clause gets lost by an application of singular DP-reduction. For Part 2 assume $\text{ld}_F(v) = 1$. Then the assertion follows from the fact, that due to the saturatedness of F we have for the clause $C \in F$ with $v \in C$ and for every clause $D \in F$ with $\bar{v} \in D$ that $C \setminus \{v\} \subseteq D \setminus \{\bar{v}\}$. \square

An example showing that in Part 1 nonsingularity of F_0 is needed, is given trivially by $F = F_0 = \{\{v\}, \{\bar{v}\}\}$. While an example for Part 2 with $F \in \mathcal{MU} \setminus \mathcal{SMU}$ and $F_0 \not\subseteq^{\mapsto} F$ is given by $F_0 = F' = \mathcal{F}_3$ (recall Definition 11) and $F = \{\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}, \{v_1, v_2, v\}, \{\bar{v}, v_3\}, \{\bar{v}_1, v_2\}, \{\bar{v}_2, v_3\}, \{\bar{v}_3, v_1\}\}$.

Theorem 16. $\mathcal{SMU} \subset \mathcal{CFMU}$.

Proof. Consider $F \in \mathcal{SMU}$ and two nonsingular $F', F'' \in \mathcal{SMU}$ with $F \xrightarrow{\text{sDP}}_* F'$ and $F \xrightarrow{\text{sDP}}_* F''$. From $F' \subseteq^{\mapsto} F$ and $F \xrightarrow{\text{sDP}}_* F'$ by Lemma 15, Part 2 we get $F' \subseteq^{\mapsto} F$, and then by Part 1 we get $F' \subseteq^{\mapsto} F''$; in the same way we obtain $F'' \subseteq^{\mapsto} F'$ and thus $F' = F''$. \square

5 Permutations of sequences of DP-reductions

This section contains central technical results on (iterated) singular DP-reduction. The basic observations are collected in Subsection 5.1, studying how literal degrees change under sDP-reductions. It follows an interlude on iterated general DP-reduction in Subsection 5.2, stating “commutativity modulo subsumption” and deriving the basic fact in Corollary 23, that in case a sequence of DP-reductions as well as some permutation both yield minimally unsatisfiable clause-sets, then actually these MU’s are the same. In Subsection 5.3 then conclusions for singular DP-reductions are drawn, obtaining various conditions under which sDP-reductions can be permuted without changing the final result. In Subsection 5.4 we introduce the “singularity index”, the minimal length of a maximal sDP-reduction sequence. Our second major result is Theorem 39, showing that in fact all maximal sDP-reduction-sequences have the same length.

5.1 Monitoring literal degrees under singular DP-reductions

First we analyse the changes for literal-degrees after one step of sDP-reduction.

Lemma 17. *Consider $F \in \mathcal{MU}$ and an m -singular variable v ($m \in \mathbb{N}$). Let C be the main clause, and let D_1, \dots, D_m be the side clauses; and let $F' := \text{DP}_v(F)$. Consider a literal $x \in \mathcal{LIT}$; the task is to compare $\text{ld}_F(x)$ and $\text{ld}_{F'}(x)$.*

1. If $\text{var}(x) \neq v$ and $x \notin C$, then $\text{ld}_{F'}(x) = \text{ld}_F(x)$.
2. If $\text{var}(x) = v$, then $\text{ld}_F(x) + \text{ld}_F(\bar{x}) = m + 1$, while $\text{ld}_{F'}(x) = \text{ld}_{F'}(\bar{x}) = 0$.
3. If $x \in C$ then $\max(m, \text{ld}_F(x) - 1) \leq \text{ld}_{F'}(x) \leq \text{ld}_F(x) - 1 + m$.
4. For $\text{var}(x) \neq v$ and $x \in C$ let $p := |\{i \in \{1, \dots, m\} : x \notin D_i\}| \in \{0, \dots, m\}$; then $\text{ld}_{F'}(x) = \text{ld}_F(x) - 1 + p$.
5. The following conditions are equivalent for $\text{var}(x) \neq v$:
 - (a) $\text{ld}_{F'}(x) = \text{ld}_F(x) - 1$.
 - (b) $\text{ld}_{F'}(x) < \text{ld}_F(x)$.
 - (c) $x \in C \cap D_1 \cap \dots \cap D_m$.
 - (d) $\text{var}(x) \in \text{var}(C) \cap \text{var}(D_1) \cap \dots \cap \text{var}(D_m)$.
6. If $\text{ld}_{F'}(x) < \text{ld}_F(x)$, then $\text{ld}_{F'}(x) \geq m$.
7. The following conditions are equivalent for $\text{var}(x) \neq v$:
 - (a) $\text{ld}_{F'}(x) > \text{ld}_F(x)$.
 - (b) $x \in C$ and $p \geq 2$ (i.e., there are at least $i \neq j$ with $x \notin D_i$ and $x \notin D_j$; so $m \geq 2$).
8. If $m = 1$, then $\text{ld}_{F'}(x) \leq \text{ld}_F(x)$.

Proof. Parts 1 - 4 follow by definition. Parts 5, 8 follows by Parts 1 - 4 and the observation, that if $x \in C$, then $\bar{x} \notin D_1 \cup \dots \cup D_m$ (due to $F \in \mathcal{MU}$). Parts 6, 7 follows by Part 5. \square

By Lemma 17, Parts 6 and 8 we get:

Corollary 18. Consider $F \in \mathcal{MU}$ and an m -singular variable v for F ($m \in \mathbb{N}$); let $F' := \text{DP}_v(F)$. Consider an arbitrary variable w .

1. “Singularity (i.e., non-1-singularity) of w can only be created for $m = 1$ ”:
 - (a) If $m \geq 2$, then w being singular for F' implies that w is singular for F ; especially if w is 1-singular for F' then w is 1-singular for F , and if w is non-1-singular for F' then w is non-1-singular for F .
 - (b) If $m = 1$ and w is singular for F' but not for F , then w is non-1-singular for F' .
2. “Singularity (non-1-singularity) of $w \neq v$ can only be destroyed for $m \geq 2$ ”:
 - (a) If $m = 1$, then w singular for F implies w singular for F' .
 - (b) If $m \geq 2$ and w is singular for F but not for F' , then w is non-1-singular for F .

By Lemma 17, Part 7 together with Lemma 9 we get:

Corollary 19. Consider $F \in \mathcal{SMU}$ and a singular variable v ; let $F' := \text{DP}_v(F)$.

1. For all literals x holds $\text{ld}_{F'}(x) \leq \text{ld}_F(x)$.
2. Thus if $w \neq v$ is a singular variable for F , then w is also singular for F' .

5.2 Iterated DP-reduction

Definition 20. Consider $F \in \mathcal{CLS}$ and a sequence v_1, \dots, v_n of variables for $n \in \mathbb{N}_0$. Then

$$\text{DP}_{v_1, \dots, v_n}(F) := \begin{cases} F & \text{if } n = 0 \\ \text{DP}_{v_n}(\text{DP}_{v_1, \dots, v_{n-1}}(F)) & \text{if } n > 0 \end{cases}.$$

Thus in “ $\text{DP}_{v_1, \dots, v_n}$ ” DP-reduction is performed in order v_1, \dots, v_n . We have $\text{var}(\text{DP}_{v_1, \dots, v_n}(F)) \subseteq \text{var}(F) \setminus \{v_1, \dots, v_n\}$. In [12] (Lemma 7.4, page 33) as well as in [13] (Lemma 7.6, page 27) the following fundamental result on iterated DP-reduction is shown:

Lemma 21. *If performing subsumption-elimination at the end, then iterated DP-reduction does not depend on the order of the variables; and performing subsumption-elimination inbetween does not influence then the result.*

More precisely, let $\text{r}_S : \mathcal{CLS} \rightarrow \mathcal{CLS}$ be subsumption-elimination, that is, $\text{r}_S(F)$ is the set of $C \in F$ which are minimal in F w.r.t. the subset-relation. And for $n \in \mathbb{N}_0$ let S_n be the set of permutations of $\{1, \dots, n\}$. Then we have the following operator-equalities for all variable-sequences $v_1, \dots, v_n \in \mathcal{VA}$ ($n \in \mathbb{N}_0$):

1. $\text{r}_S \circ \text{DP}_{v_1, \dots, v_n} = \text{r}_S \circ \text{DP}_{v_1, \dots, v_n} \circ \text{r}_S$.
2. For all $\pi \in S_n$ we have $\text{r}_S \circ \text{DP}_{v_1, \dots, v_n} = \text{r}_S \circ \text{DP}_{v_{\pi(1)}, \dots, v_{\pi(n)}}$.

Definition 22. Consider $F \in \mathcal{CLS}$ and $v_1, \dots, v_n \in \mathcal{VA}$ ($n \in \mathbb{N}_0$). Then a permutation $\pi \in S_n$ is called **equality-preserving** for F and v_1, \dots, v_n (for short: “eq-preserving”), if we have $\text{DP}_{v_1, \dots, v_n}(F) = \text{DP}_{v_{\pi(1)}, \dots, v_{\pi(n)}}(F)$. The set of all eq-preserving $\pi \in S_n$ is denoted by $\text{eqp}(F, (v_1, \dots, v_n)) \subseteq S_n$.

Note that if $\text{var}(F) \subseteq \{v_1, \dots, v_n\}$, then $\text{eqp}(F, (v_1, \dots, v_n)) = S_n$. Since minimally unsatisfiable clause-sets do not contain subsumptions, we obtain:

Corollary 23. Consider $F \in \mathcal{CLS}$ and variables v_1, \dots, v_n ($n \in \mathbb{N}_0$) such that $\text{DP}_{v_1, \dots, v_n}(F) \in \mathcal{MU}$. Then we have for $\pi \in S_n$:

$$\pi \in \text{eqp}(F, (v_1, \dots, v_n)) \Leftrightarrow \text{DP}_{v_{\pi(1)}, \dots, v_{\pi(n)}}(F) \in \mathcal{MU}.$$

Since hitting clause-sets do not contain subsumptions, by Lemma 2 we obtain:

Corollary 24. For clause-sets $F \in \mathcal{HIT}$ and variables v_1, \dots, v_n ($n \in \mathbb{N}_0$) we have $\text{eqp}(F, (v_1, \dots, v_n)) = S_n$.

5.3 Iterated sDP-reduction via singular tuples

Generalising Definition 4 we consider “singular tuples”:

Definition 25. Consider $F \in \mathcal{MU}$. A tuple (v_1, \dots, v_n) of variables ($n \in \mathbb{N}_0$) is called **singular** for F if for all $i \in \{1, \dots, n\}$ we have that v_i is singular for $\text{DP}_{v_1, \dots, v_{i-1}}(F)$. Note that for a singular (v_1, \dots, v_n) all variables must be different. We call variable v_i ($i \in \{1, \dots, n\}$) **m-singular** ($m \in \mathbb{N}$) for (v_1, \dots, v_n) and F , if v_i is m-singular for $\text{DP}_{v_1, \dots, v_{i-1}}(F)$.

Understanding for $F \in \mathcal{MU}$ the set of singular tuples is a fundamental task for the classification of \mathcal{MU} . Two basic properties are:

1. F has only the empty singular tuple iff F is nonsingular.
2. If (v_1, \dots, v_n) is a singular tuple for F , then for all $i \in \{0, \dots, n\}$ the tuple (v_1, \dots, v_i) is also singular for F .

Definition 26. Consider $F \in \mathcal{MU}$ and a singular tuple (v_1, \dots, v_n) for F . A permutation $\pi \in S_n$ is called **singularity-preserving** for F and (v_1, \dots, v_n) (for short: “s-preserving”), if also $(v_{\pi(1)}, \dots, v_{\pi(n)})$ is singular for F . The set of all s-preserving $\pi \in S_n$ is denoted by $\text{sp}(F, (v_1, \dots, v_n)) \subseteq S_n$.

By Corollary 23 we obtain the fundamental lemma:

Lemma 27. *For $F \in \mathcal{MU}$ and a singular tuple \mathbf{v} we have $\text{sp}(F, \mathbf{v}) \subseteq \text{eqp}(F, \mathbf{v})$.*

Corollary 28. *Consider two singular tuples $(v_1, \dots, v_n), (v'_1, \dots, v'_n)$ for $F \in \mathcal{MU}$. If $\{v_1, \dots, v_n\} = \{v'_1, \dots, v'_n\}$, then $\text{DP}_{v_1, \dots, v_n}(F) = \text{DP}_{v'_1, \dots, v'_n}(F)$.*

In preparation for our results on singularity-preserving permutation, we consider first “homogeneous” singular pairs in the following two lemmas. By Lemma 17, Parts 3 and 6 we get:

Lemma 29. *Consider $F \in \mathcal{MU}$ and two different non-1-singular variables v, w for F . Let C be the main clause for v , and let D be the main clause for w . There are precisely two cases now:*

1. *If $C = D$, then $w \notin \text{var}_s(\text{DP}_v(F))$ and $v \notin \text{var}_s(\text{DP}_w(F))$.*
2. *If $C \neq D$, then $w \in \text{var}_{-1s}(\text{DP}_v(F))$ and $v \in \text{var}_{-1s}(\text{DP}_w(F))$.*

Lemma 30. *Consider $F \in \mathcal{MU}$ and a singular sequence (v, w) for F , where both v and w are 1-singular (that is, in F resp. $\text{DP}_v(F)$). Let $C, D \in F$ be the two occurrences of v .*

1. *Assume w is not 1-singular in F :*
 - (a) *Then w is 2-singular in F . Let $E_0 \in F$ be the main-clause of w , and let $E_1, E_2 \in F$ be the two side-clauses.*
 - (b) *We have $\{E_1, E_2\} = \{C, D\}$.*
 - (c) *So v is 1-singular in $\text{DP}_w(F)$.*
2. *Otherwise w is 1-singular in F .*
 - (a) *v is 1-singular in $\text{DP}_w(F)$.*
 - (b) *Let E_1, E_2 be the two occurrences of w in F : $|\{C, D\} \cap \{E_1, E_2\}| \leq 1$.*

Now we are ready to give the central lemma on singularity-preserving neighbour-exchanges (note that every permutation is the composition of neighbour-exchanges):

Lemma 31. *Consider $F \in \mathcal{MU}$ and a singular tuple $\mathbf{v} = (v_1, \dots, v_n)$ with $n \geq 2$. Consider $i \in \{1, \dots, n-1\}$, and let $\pi \in S_n$ be the neighbour-exchange $i \leftrightarrow i+1$ (i.e., $\pi(j) = j$ for $j \in \{1, \dots, n\} \setminus \{i, i+1\}$, while $\pi(i) = i+1$ and $\pi(i+1) = i$). The task is to characterise when $\pi \in \text{sp}(F, \mathbf{v})$ holds; we need also to be able to apply such s -preserving neighbour-exchanges consecutively. For this let v_i be m_i -singular w.r.t. F, \mathbf{v} , and m'_i -singular w.r.t. F, \mathbf{v}' , where $\mathbf{v}' := (v_{\pi(1)}, \dots, v_{\pi(n)})$, in case of $\pi \in \text{sp}(F, \mathbf{v})$.*

1. *If $\pi \in \text{sp}(F, \mathbf{v})$, then for $j \in \{1, \dots, n\} \setminus \{i, i+1\}$ we have $m'_j = m_j$.*
2. *Assume $m_i \geq 2$.*
 - (a) *$\pi \in \text{sp}(F, \mathbf{v})$.*
 - (b) *If $m_{i+1} \geq 2$, then $m'_i, m'_{i+1} \geq 2$.*
 - (c) *If $m_{i+1} = 1$, then $m'_i = 1$ and $m'_{i+1} \geq m_i - 1$ (where $m'_{i+1} = 1$ as well as $m'_{i+1} \geq 2$ are possible in general).*
3. *Assume $m_i = 1$.*
 - (a) *Assume $m_{i+1} = 1$.*
 - i. *$\pi \in \text{sp}(F, \mathbf{v})$.*
 - ii. *$m'_{i+1} = 1$ and $m'_i \in \{1, 2\}$.*
 - (b) *Assume $m_{i+1} \geq 2$.*

- i. $\pi \in \text{sp}(F, \mathbf{v})$ if and only if v_{i+1} is singular in $\text{DP}_{v_1, \dots, v_{i-1}}(F)$.
- ii. If $\pi \in \text{sp}(F, \mathbf{v})$, then $m'_i \geq 2$ (while $m'_{i+1} = 1$ and $m'_{i+1} \geq 2$ is possible in general).

Proof. Part 1 follows by Lemma 27. For the remainder let $F_0 := F$, and $F_i := \text{DP}_{v_i}(F_{i-1})$ for $i \in \{1, \dots, n\}$. Consider Part 2; so we assume $m_i \geq 2$ here. By Corollary 18, Part 1, we get that v_{i+1} is singular for F_{i-1} . In case of $m_{i+1} \geq 2$, that is, Part 2b, we have Case 2 of Lemma 29 here, and the assertions of Parts 2a, 2b follow. In case of $m_{i+1} = 1$, that is, Part 2c, we get the assertions of Parts 2a, 2c by Parts 6, 3 of Lemma 17. Finally consider Part 3, assuming $m_i = 1$. Part 3a follows with Lemma 30. For Part 3b assume $m_{i+1} \geq 2$. For Part 3(b)i the direction from left to right follows by definition, while the direction from right to left follows by Part 2b of Lemma 18. And Part 3(b)ii by Part 8 of Lemma 17. \square

Corollary 32. Consider $F \in \mathcal{MU}$ and a singular tuple \mathbf{v} such that each v_i is non-1-singular for \mathbf{v} . Then $\text{sp}(F, \mathbf{v}) = S_n$ (all permutations are singular), and in each permutation of \mathbf{v} each v_i is non-1-singular.

Corollary 33. Consider $F \in \mathcal{MU}$ and a singular tuple \mathbf{v} such that each v_i is 1-singular in F . Then $\text{sp}(F, \mathbf{v}) = S_n$ (all permutations are singular), and in each permutation of \mathbf{v} each v_i is 1-singular.

We get some normal form by moving the 1-sDP-reductions to the front:

Corollary 34. Consider $F \in \mathcal{MU}$ and a singular tuple (v_1, \dots, v_n) . Then there is a singular permutation π , such that in $(v_{\pi(1)}, \dots, v_{\pi(n)})$ all 1-singular variables are before all non-1-singular variables. Furthermore the initial segment consists of the v_i which are 1-singular variables in F , and every such v_i can come first.

If $F \in \mathcal{MU}$ has no 1-singular variables, then we know its singular tuples:

Lemma 35. Consider $F \in \mathcal{MU}$ with $\text{var}_{1s}(F) = \emptyset$.

- By Corollary 18, Part 1a, for each singular tuple (v_1, \dots, v_n) of F we have $\{v_1, \dots, v_n\} \subseteq \text{var}_{-1s}(F)$, and every v_i is non-1-singular w.r.t. (v_1, \dots, v_n) .
- So by Corollary 32 all permutations are singular ($\text{sp}(F, (v_1, \dots, v_n)) = S_n$).

Thus the singular tuples of F are fully characterised by the subsets $S \subseteq \text{var}_{-1s}(F)$ such that there exists a linear order of S which yields a singular tuple for F ; let $\mathbb{S}_F \subseteq \mathbb{P}(\text{var}_{-1s}(F))$ be the set of all such subsets (we obtain precisely all singular tuples of F by taking all linear orders of $S \in \mathbb{S}_F$).

For $v \in \text{var}_{-1s}(F)$ let x_v be the corresponding singular literal (that is, $\text{var}(x_v) = v$ and $\text{ld}_F(x_v) = 1$), and let $L_V := \{x_v : v \in \text{var}_{-1s}(F)\}$. And let $F'_F := \{C \cap L_F : C \in F \wedge C \cap L_F \neq \emptyset\}$ be the set of sub-clauses given by the singular literals in main clauses of F . Finally let the hypergraph $G_F := (\text{var}_{-1s}(F), \{\text{var}(C) : C \in F'_F\})$ be given by the singular variables in the main clauses. Note that the hyperedges of G_F are non-empty and pairwise disjoint.

Now \mathbb{S}_F is the set of partial exact transversals of G_F , that is, subsets $S \subseteq \text{var}_{-1s}(F)$ which have with each hyperedge of G_F at most one element in common, that is, $\forall H \in E(G_F) : |S \cap H| \leq 1$.

Proof. For $v \in \text{var}_{-1s}(F)$ let $F_v := \text{DP}_v(F)$, let $C_v \in F$ be the main clause of v , and let $H_v := \text{var}(C_v) \cap \text{var}_{-1s}(F)$. Then we have $G_{F_v} = (V(G_F) \setminus H_v, E(G_F) \setminus \{H_v\})$. The assertion of the lemma follows now easily by induction. \square

Comparing two different singular tuples, they don't need to overlap, however they need to have a commuting beginning via appropriate permutations, given they contain at least two variables:

Lemma 36. *Consider $F \in \mathcal{MU}$ and two singular tuples (v_1, \dots, v_p) , (w_1, \dots, w_q) for F with $p, q \geq 2$. Then there exists a singular permutation π for (v_1, \dots, v_p) and a singular permutation π' for (w_1, \dots, w_q) such that both $(v_{\pi(1)}, w_{\pi'(1)})$ and $(w_{\pi'(1)}, v_{\pi(1)})$ are singular for F .*

Proof. If one of the two tuples contains a 1-singular variable v_i resp. w_i , then the assertion follows by Corollary 34 and Part 2 of Corollary 18. And otherwise the assertion follows by Corollary 32 and Lemma 29. \square

5.4 The singularity index

Definition 37. *Consider $F \in \mathcal{MU}$. A singular tuple (v_1, \dots, v_n) for F is called **maximal**, if there is no singular tuple extending it (i.e., $\text{DP}_{v_1, \dots, v_n}(F)$ is non-singular). The **singularity index** of F , denoted by $\text{si}(F) \in \mathbb{N}_0$, is the minimal $n \in \mathbb{N}_0$ such that a maximal singular sequence of length n exists for F .*

So $\text{si}(F) = 0 \Leftrightarrow F \in \mathcal{MU}'$. See Corollary 45, Part 1, for a characterisation of $F \in \mathcal{MU}$ with $\text{si}(F) = 1$. In Theorem 39 we see that all maximal singular tuples are of the same length (given by the singularity index). By Lemma 35 we get:

Lemma 38. *Consider $F \in \mathcal{MU}$ not having 1-singular variables (i.e., $\text{var}_{1s}(F) = \emptyset$). Then every maximal singular tuple has length $\text{si}(F)$, which is the number of different clauses of F containing at least one singular literal.*

More general than Lemma 35 (but with less details):

Theorem 39. *For $F \in \mathcal{MU}$ and every maximal singular tuple (v_1, \dots, v_m) for F we have $m = \text{si}(F)$.*

Proof. We prove the assertion by induction on $\text{si}(F)$. For $\text{si}(F) = 0$ the assertion is trivial, so assume $\text{si}(F) > 0$. If F has no 1-singular variables, then the assertion follows by Lemma 35, and so we assume that F has a 1-singular variable v . First we show that we can choose v such that $\text{si}(\text{DP}_v(F)) = n - 1$.

Consider a maximal singular tuple (v_1, \dots, v_n) of length $n = \text{si}(F)$. Note that $\text{si}(\text{DP}_{v_1}(F)) = n - 1$. If v_1 is 1-singular, then we are done, and so assume v_1 is not 1-singular. Now by Lemma 18, Part 2, both tuples (v_1, v) and (v, v_1) are singular for F , where by induction hypothesis we have $\text{si}(\text{DP}_{v_1, v}(F)) = n - 2$. Due to $\text{DP}_{v_1, v}(F) = \text{DP}_{v, v_1}(F)$ we get then (using the induction hypothesis) $\text{si}(\text{DP}_v(F)) = n - 1$ as claimed.

Now consider an arbitrary maximal singular tuple (w_1, \dots, w_m) . We have to show that $\text{si}(\text{DP}_{w_1}(F)) = n - 1$, from which by induction hypothesis the assertion follows. This follows by the same argument as above, but for completeness we repeat it. The claim holds for $w_1 = v$, and so assume $w_1 \neq v$. Then we repeat it. The claim holds for $w_1 = v$, and so assume $w_1 \neq v$. Now by Corollary 18, Part 2, both tuples (v, w_1) and (w_1, v) are singular for F , where by induction hypothesis we have $\text{si}(\text{DP}_{v, w_1}(F)) = n - 2$. Due to $\text{DP}_{v, w_1}(F) = \text{DP}_{w_1, v}(F)$ we get then (using the induction hypothesis) $\text{si}(\text{DP}_{w_1}(F)) = n - 1$ as claimed. \square

Corollary 40. *For $F \in \mathcal{MU}$ and $F', F'' \in \text{sDP}(F)$ we have $n(F') = n(F'')$.*

6 Confluence mod isomorphism on eventually \mathcal{SMU}

Finally we are able to show our third major result, confluence modulo isomorphism of singular DP-reduction in case all maximal sDP-reductions yield saturated clause-sets.

Definition 41. A minimally unsatisfiable clause-set F is called *eventually saturated*, if all nonsingular F' with $F \xrightarrow{sDP}_* F'$ are saturated; the set of all eventually saturated clause-sets is $\mathcal{ESMU} := \{F \in \mathcal{MU} : \text{sDP}(F) \subseteq \mathcal{SMU}\}$.

By Corollary 10 we have $\mathcal{SMU} \subseteq \mathcal{ESMU}$. If $\mathcal{C} \subseteq \mathcal{MU}$ is stable under sDP-reduction, then we have $\mathcal{C} \subseteq \mathcal{ESMU}$ iff $\mathcal{C} \cap \mathcal{MU}' \subseteq \mathcal{SMU}$. In order to show $\mathcal{ESMU} \subseteq \mathcal{CFIMU}$ we show first that “divergence in one step” is enough:

Lemma 42. Consider $F \in \mathcal{MU} \setminus \mathcal{CFIMU}$ (recall Definition 13). So $\text{si}(F) \geq 1$. Then there is a singular tuple $(v_1, \dots, v_{\text{si}(F)-1})$ for F , such that for $F' := \text{DP}_{v_1, \dots, v_{\text{si}(F)-1}}(F)$ we still have $\text{sDP}(F') \in \mathcal{MU} \setminus \mathcal{CFIMU}$ (note $\text{si}(F') = 1$).

Proof. We prove the assertion by induction on $\text{si}(F)$. The assertion is trivial for $\text{si}(F) \leq 1$, and so assume $n := \text{si}(F) \geq 2$. We use proof by contradiction, and so assume that for all singular variables v we have $\text{DP}_v(F) \in \mathcal{CFIMU}$. Consider (maximal) singular tuples $(v_1, \dots, v_n), (w_1, \dots, w_n)$ for F such that $\text{DP}_v(F)$ and $\text{DP}_w(F)$ are not isomorphic. By Lemma 36 w.l.o.g. we can assume that (v_1, w_1) and (w_1, v_1) are both singular for F . Since $\text{DP}_{v_1}(F), \text{DP}_{w_1}(F) \in \mathcal{CFIMU}$, we obtain the contradiction that $\text{DP}_v(F)$ and $\text{DP}_w(F)$ are isomorphic. \square

Corollary 43. Consider a class $\mathcal{C} \subseteq \mathcal{MU}$ which is stable under application of singular DP-reduction. Then we have $\mathcal{C} \subseteq \mathcal{CFIMU}$ if and only if $\{F \in \mathcal{C} : \text{si}(F) = 1\} \subseteq \mathcal{CFIMU}$.

Now we analyse the main case where all sDP-reductions give saturated results:

Lemma 44. Consider $F \in \mathcal{MU}$ and a clause $C \in F$. Let $C' := \{x \in C : \text{ld}_F(x) = 1\}$ be the set of singular literals in C , establishing C as the main clause for the underlying singular variables $\text{var}(x)$ (for $x \in C'$), and let $F_x := \{D \in F : \bar{x} \in D\}$ be the set of side clauses of $\text{var}(x)$ for $x \in C'$. Due to $F \in \mathcal{MU}$ the sets F_x are non-empty and pairwise disjoint (note that $\text{var}(x)$ is $|F_x|$ -singular in F for $x \in C'$). Now assume $|C'| \geq 2$, and that for all $x \in C'$ we have $\text{DP}_{\text{var}(x)}(F) \in \mathcal{SMU}$. Then:

1. $|C'| = 2$.
2. $\forall x \in C' \forall D \in F_x : (C \setminus C') \subseteq D$.
3. For $x, y \in C'$ we have that $\text{DP}_{\text{var}(x)}(F)$ and $\text{DP}_{\text{var}(y)}(F)$ are isomorphic.

Proof. Consider (any) literals $x, y \in C'$ with $x \neq y$. Then for $D \in F_x$ we have $(C \setminus \{x, y\}) \subseteq D$ by Corollary 8, since otherwise the corollary can be applied to $\text{var}(x)$, replacing D by $D \cup (C \setminus \{x, y\})$, which yields the partial saturation $F' \in \mathcal{MU}$ of F with singular variable $\text{var}(y)$, and where then $\text{DP}_{\text{var}(y)}(F')$ would yield a proper partial saturation G of $\text{DP}_{\text{var}(y)}(F)$, contradicting that the latter is saturated. It follows that actually $C' = \{x, y\}$ must be the case, since if there would be $z \in C' \setminus \{x, y\}$, then $\text{ld}_F(z) \geq 2$ contradicting the definition of C' . It follows Part 2. Finally for Part 3 we note that now $F \rightsquigarrow \text{DP}_x(F)$ just replaces \bar{x} in the clauses of F_x by y , while $F \rightsquigarrow \text{DP}_y(F)$ just replaces \bar{y} in the clauses of F_y by x , and thus renaming y in $\text{DP}_x(F)$ to \bar{x} yields $\text{DP}_y(F)$. \square

Corollary 45. *For $F \in \mathcal{MU}$ with $\text{si}(F) = 1$ we have:*

1. *If $|\text{var}_s(F)| \geq 2$:*
 - (a) *$\text{var}_s(F) = \text{var}_{\neg 1s}(F)$, that is, all singular variables are non-1-singular.*
 - (b) *The main clauses of the singular variables coincide (that is, there is $C \in F$ such that for all singular literals x for F we have $x \in C$).*
 - (c) *If $F \in \mathcal{ESMU}$ then $|\text{var}_s(F)| = 2$.*
2. *If $F \in \mathcal{ESMU}$ then $F \in \mathcal{CFIMU}$.*

Proof. Part 1a follows by Part 2a of Corollary 18, and Part 1b follows by Lemma 29. Now Parts 1c, 2 follow from Lemma 44. \square

By Corollary 43 we obtain from Part 2 of Corollary 45:

Theorem 46. $\mathcal{ESMU} \subset \mathcal{CFIMU}$.

6.1 Applications to $\mathcal{MU}_{\delta=2}$

Consider $k \in \mathbb{N}$ and $F \in \mathcal{MU}_{\delta=k}$. Assume that we know the isomorphism types of $\mathcal{MU}'_{\delta=k}$. If $F \in \mathcal{CFIMU}$, then we can speak of *the type* of F as the (unique) isomorphism type of the elements of $\text{sDP}(F)$. As an application of our results, we obtain now that for (arbitrary) $F \in \mathcal{MU}_{\delta=2}$ we can speak of the type of F as the number of variables left after complete sDP-reduction (using that the isomorphism types in $\mathcal{MU}_{\delta=2}$ are determined by their number of variables):

Theorem 47. $\mathcal{MU}_{\delta=2} \subseteq \mathcal{CFIMU}$.

Proof. Recall Theorem 12. The first proof is obtained by applying the singularity index, Corollary 40, and the observation that non-isomorphic elements of $\mathcal{MU}'_{\delta=2}$ have different numbers of variables. The second proof is obtained by applying Theorem 46 and the fact that $\mathcal{MU}'_{\delta=2} \subseteq \mathcal{SMU}$, whence $\mathcal{MU}_{\delta=2} \subseteq \mathcal{ESMU}$. \square

7 Conclusion and open problems

We have discussed questions regarding confluence of singular DP-reduction on minimally unsatisfiable clause-sets. Besides various detailed characterisations, we obtained the invariance of the length of maximal sDP-reduction-sequences, confluence for saturated and confluence modulo isomorphism for eventually saturated clause-sets. The main open questions regarding these aspects are:

- Are there other interesting classes for which we can show confluence resp. confluence mod isomorphism of singular DP-reduction?
- Can we characterise \mathcal{CFMU} and/or \mathcal{CFIMU} ? Especially, what is the decision complexity of these classes?

As a first application of our results, in Subsection 6.1 we considered the types of (arbitrary) elements of $\mathcal{MU}_{\delta=2}$. This detailed knowledge is an important stepping stone for the determination of the isomorphism types of the elements of $\mathcal{MU}'_{\delta=3}$, which we have obtained meanwhile (to be published; based on a mixture of general insights into the structure of \mathcal{MU} and (very) detailed investigations into $\mathcal{MU}_{\delta \leq 2}$). The major open problem of the field is the classification (of isomorphism types) of $\mathcal{MU}'_{\delta=k}$ for arbitrary k . Finally, regarding the potential applications from Subsection 1.3, applying singular DP-reductions in algorithms searching for MUS's is a natural next step.

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